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The algebraic structure of zero curvature representations and application to coupled KdV systems

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Abstract. We first establish an algebraic structure related to zero curvature representations and propose a new approach for calculating symmetry algebras of integrable systems. Then we deduce a hierarchy of non-isospectral flows associated with coupled KdV systems from a spectral problem with the Laurent polynomial dependent form of the spectral parameter. Furthermore, the commutator relations of Lax operators corresponding to isospectral and non-isospectral flows are worked out according to this algebraic structure, and thus a symmetry algebra for coupled KdV systems is engendered from this general theory.

1. Introduction

In general, for a spectral problem, the non-isospectral ($\lambda_t = \lambda^n, n \geqslant 0$) hierarchy of vector fields is the usual first-order master symmetries [1] of the isospectral ($\lambda_t = 0$) hierarchy of evolution equations. Moreover, isospectral and non-isospectral flows often constitute a semi-product Lie algebra of a Kac–Moody algebra and a Virasoro algebra (see, for example, [2–5]). Furthermore, on the basis of Lax representations, we have shown in Ma [6] that the hierarchies of two corresponding Lax operators also constitute the same infinite-dimensional Lie algebra, when integrable systems possess a strong [7] and hereditary [8] symmetry operator.

In the present paper, we employ zero curvature representations other than Lax representations to consider integrable systems. We first propose an algebraic structure for zero curvature representations of integrable systems. Then we briefly derive the isospectral and non-isospectral hierarchies of integrable systems from a spectral problem with the Laurent polynomial dependent form of λ

$$\psi_{xx} + Q\psi = 0 \qquad Q = Q(u,\lambda) = \lambda^{-l}\sum_{i=0}^{q} v_i\lambda^i \qquad v_q = -1 \qquad q \geqslant 1$$
$$0 \leqslant l \leqslant q-1 \tag{1.1}$$

where $u = (v_0, v_1, \ldots, v_{q-1})^T$. The corresponding isospectral hierarchy has been discussed in Ma [9] and Boiti *et al* [10] in considerable detail and is called a coupled KdV hierarchy in Antonowicz and Fordy [11] because of certain coupled KdV flows of the hierarchy. Finally we show that two hierarchies of Lax operators associated with zero curvature representations constitute an infinite-dimensional Lie algebra, and thus the symmetry algebra of coupled KdV hierarchy is deduced.

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2. Algebraic structure of Lax operators of zero curvature representations

Let $x, t \in R$, $u = (u_1, u_2, \ldots, u_q)^T$, $u_i = u_i(x,t)$, $1 \leqslant i \leqslant q$. We denote by $\mathcal{B}$ all complex (or real) functions $P[u] = P(x,t,u)$ which are C^∞-differentiable with respect to x, t and C^∞-Gateaux differentiable with respect to $u = u(x)$ (as functions of x), and set $\mathcal{B}^r = \{(P_1, \ldots, P_r)^T \mid P_i \in \mathcal{B}\}$. Moreover by $\mathcal{V}^r_{(0)}$ we mean all matrix differential operators $W = (W_{ij})_{r\times r}$ of zero order, where $W_{ij} \in \mathcal{B}$, $1 \leqslant i, j \leqslant r$, and set $\widetilde{\mathcal{V}}^r_{(0)} = \mathcal{V}^r_{(0)} \otimes C[\lambda, \lambda^{-1}]$. For $K \in \mathcal{B}^r$, $W \in \widetilde{\mathcal{V}}^r_{(0)}$, and $S \in \mathcal{B}^q$ we define

$$K'[S] = \frac{\partial}{\partial\epsilon} K(u+\epsilon S)|_{\epsilon=0} \qquad W'[S] = \frac{\partial}{\partial\epsilon} W(u+\epsilon S)|_{\epsilon=0}. \tag{2.1}$$

It is known (see [12]) that $\mathcal{B}^q$ forms a Lie algebra under the following product

$$[K, S] = K'[S] - S'[K] \qquad K, S \in \mathcal{B}^q. \tag{2.2}$$

The commutator of two smooth functions $f, g \in C^\infty(C)$ (as vector fields over C) is defined as

$$[\![f, g]\!](\lambda) = f'(\lambda)g(\lambda) - f(\lambda)g'(\lambda) \qquad \lambda \in C. \tag{2.3}$$

In fact for $f, g, h \in C^\infty(C)$, we easily see that

$$\begin{aligned}[\![[\![f, g]\!], h]\!] + \text{cycle } (f,g,h) &= [\![f'g - fg', h]\!] + \text{cycle } (f,g,h) \\ &= (f'g - fg')'h - (f'g - fg')h' + \text{cycle } (f,g,h) \\ &= (f''gh - fg''h) - (f'gh' - fg'h') + \text{cycle } (f,g,h) = 0\end{aligned}$$

and thus the bracket (2.3) indeed defines a Lie algebra structure over $C^\infty(C)$.

In what follows, we always assume that the spectral operator $U = U(u,\lambda) \in \widetilde{\mathcal{V}}^r_{(0)}$ has an injective Gateaux derivative operator $U' : \mathcal{B}^q \to \widetilde{\mathcal{V}}^r_{(0)}$. We consider the spectral problem

$$\phi_x = U\phi = U(u,\lambda)\phi \qquad \lambda_t = f(\lambda) \qquad \phi_t = V\phi = V(u,\lambda)\phi \tag{2.4}$$

where $V \in \widetilde{\mathcal{V}}^r_{(0)}$, $f \in C^\infty(C)$. Noticing that $U_t = U'[u_t] + \lambda_t U_\lambda$, $U_\lambda = \partial U/\partial\lambda$, we obtain that the evolution equation $u_t = K = K(x,t,u)$, $K \in \mathcal{B}^q$, possesses a zero curvature representation corresponding to (2.4)

$$U_t - V_x + [U, V] = 0 \tag{2.5}$$

if and only if the vector field K satisfies

$$U'[K] + f(\lambda)U_\lambda - V_x + [U, V] = 0. \tag{2.6}$$

Definition 2.1. Let $V \in \widetilde{\mathcal{V}}^r_{(0)}$. If there exist $K \in \mathcal{B}^q$ and $f \in C^\infty(C)$ such that (2.6) holds, then V is called a Lax operator corresponding to f and K is called an eigenvector field of V corresponding to f.

Assume that $\mathcal{P}(U)$ denotes all triples $(V, K, f) \in \widetilde{\mathcal{V}}^r_{(0)} \times \mathcal{B}^q \times C^\infty(C)$ satisfying (2.6) and, for $f \in C^\infty(C)$, we set

$$\mathcal{M}(U, f) = \{V \in \widetilde{\mathcal{V}}^r_{(0)} | \exists K \in \mathcal{B}^q \text{ so that } (V, K, f) \in \mathcal{P}(U)\}$$

i.e. all Lax operators corresponding to f, and

$$E\mathcal{M}(U, f) = E_f\mathcal{M}(U, f) = \{K \in \mathcal{B}^q | \exists V \in \mathcal{M}(U, f) \text{ so that } (V, K, f) \in \mathcal{P}(U)\}$$

i.e. all eigenvector fields of $\mathcal{M}(U, f)$ corresponding to f. For $(V, K, f), (W, S, g) \in \mathcal{P}(U)$, we construct a new operator $[\![V, W]\!] \in \widetilde{\mathcal{V}}^r_{(0)}$ as follows

$$[\![V, W]\!] = V'[S] - W'[K] + [V, W] + gV_\lambda - fW_\lambda \tag{2.7}$$

which plays a crucial role in our theory.

Theorem 2.1. Let (V, K, f), $(W, S, g) \in \mathcal{P}(U)$. Then $([\![V, W]\!], [K, S], [\![f, g]\!])$ belongs to $\mathcal{P}(U)$, too, i.e.

$$U'[[K, S]] + [\![f, g]\!](\lambda)U_\lambda - [\![V, W]\!]_x + [U, [\![V, W]\!]] = 0 \tag{2.8}$$

which implies that the operator $[\![V, W]\!]$ defined by (2.7) is a Lax operator corresponding to $[\![f, g]\!]$ given by (2.3).

Proof. By using the assumption (V, K, f), $(W, S, g) \in \mathcal{P}(U)$, we obtain

$$\begin{aligned} &V_x'[S] - [U, V]'[S] = (U'[K])'[S] + fU_\lambda'[S] \\ &W_x'[K] - [U, W]'[K] = (U'[S])'[K] + gU_\lambda'[K] \end{aligned} \tag{2.9a}$$

$$\begin{aligned} &U_\lambda'[K] = V_{x\lambda} - [U, V]_\lambda - f_\lambda U_\lambda - fU_{\lambda\lambda} \\ &U_\lambda'[S] = W_{x\lambda} - [U, W]_\lambda - g_\lambda U_\lambda - gU_{\lambda\lambda}. \end{aligned} \tag{2.9b}$$

Set $\Xi = V'[S] - W'[K] + [V, W]$. We have

$$\Xi_x - [U, \Xi] = V_x'[S] - W_x'[K] + [V_x, W] + [V, W_x] - [U, V'[S] - W'[K] + [V, W]].$$

In addition,

$$[U, [V, W]] = [V, [U, W]] - [W, [U, V]] = [V, W_x - U'[S] - gU_\lambda] - [W, V_x - U'[K] - fU_\lambda].$$

Therefore by (2.9*a*),(2.9*b*),

$$\begin{aligned} \Xi_x - [U, \Xi] &= V_x'[S] - W_x'[K] - [U, V'[S] - W'[K]] \\ &\quad + [V, U'[S] + gU_\lambda] - [W, U'[K] + fU_\lambda] \\ &= V_x'[S] - W_x'[K] - [U, V]'[S] + [U, W]'[K] + g[V, U_\lambda] - f[W, U_\lambda] \\ &= U'[[K, S]] + fU_\lambda'[S] - gU_\lambda'[K] + g[V, U_\lambda] - f[W, U_\lambda] \qquad \text{(by (2.9}a\text{))} \\ &= U'[[K, S]] + fW_{x\lambda} - f[U, W_\lambda] - gV_{x\lambda} + g[U, V_\lambda] + [\![f, g]\!]\, U_\lambda \qquad \text{(by (2.9}b\text{))}. \end{aligned} \tag{2.10}$$

It follows from the equality (2.10) that $([\![V, W]\!], [K, S], [\![f, g]\!])$ satisfies (2.6) and thus the proof is completed. □

By the above theorem, we easily know that if two evolution equations $u_t = K, u_t = S$ $(K, S \in \mathcal{B}^q)$ are, respectively, the compatibility conditions of the spectral problems

$$\begin{aligned} &\phi_x = U\phi \qquad \lambda_t = a\lambda^m \qquad \phi_x = v\phi \qquad v \in \widetilde{\mathcal{V}}_{(0)}^r \\ &\phi_t = U\phi \qquad \lambda_t = b\lambda^h \qquad \phi_t = W\phi \qquad W \in \widetilde{\mathcal{V}}_{(0)}^r \end{aligned}$$

where $a, b =$ constants, $m, n \geqslant 0$, then the product equation $u_t = [K, S]$ is the compatibility condition of the following spectral problem

$$\phi_x = U\phi \qquad \lambda_t = ab(m - n)\lambda^{m+n-1} \qquad \phi_t = [\![V, W]\!]\phi$$

with $[\![V, W]\!] = V'[S] - W'[K] + [V, W] + b\lambda^n V_\lambda - a\lambda^m W_\lambda$. Hence we see that

$$[\![\mathcal{M}(U,0), \mathcal{M}(U,\lambda^m)]\!] \subseteq \mathcal{M}(U,0) \qquad m \geqslant 0.$$

Further we have

$$[\![\mathcal{M}(U,0), [\![\mathcal{M}(U,0), \mathcal{M}(U,\lambda^m)]\!]]\!] = 0 \qquad m \geqslant 0$$

and thus

$$[E\mathcal{M}(U,0), [E\mathcal{M}(U,0), E\mathcal{M}(U,\lambda^m)]] = 0 \qquad m \geqslant 0$$

provided that $\mathcal{M}(U,0)$ is commutative. This reveals why the non-isospectral vector fields can become the first-order master symmetries of the isospectral flows. This is usually shown by complicated calculation. In addition, through using theorem 2.1 we can obtain the following two results.

Corollary 2.1. If $(V_i, K_i, f_i) \in \mathcal{P}(U)$, $i = 1, 2, 3$. Then

$$[\![[\![V_1, V_2]\!], V_3]\!]_x - [U, [\![[\![V_1, V_2]\!], V_3]\!]] + \text{cycle } (V_1, V_2, V_3) = 0.$$

Proof. Notice that $\langle \mathcal{B}^q, [\cdot,\cdot]\rangle$ and $\langle C^\infty(C), [\![\cdot,\cdot]\!]\rangle$ are Lie algebras. The required result follows theorem 2.1, namely (2.8). □

Corollary 2.2. Let $(V_i, K_i, f_i) \in \mathcal{P}(U)$, $i = 1, 2, 3$. If $[\![V_1, V_2]\!] = V_3$ and $[\![f_1, f_2]\!] = f_3$, then $[K_1, K_2] = K_3$.

Proof. By theorem 2.1, namely (2.8), and the assumption, we have

$$\begin{aligned} U'[[K_1, K_2]] &= -[\![f_1, f_2]\!](\lambda)U_\lambda + [\![V_1, V_2]\!]_x - [U, [\![V_1, V_2]\!]] \\ &= -f_3(\lambda)U_\lambda + V_{3x} - [U, V_3] = U'[K_3]. \end{aligned}$$

Thus $[K_1, K_2] = K_3$ as U' is injective. □

Because we assume that U' is injective, a Lax operator in $\mathcal{M}(U,0)$ has only an eigenvector field corresponding to $f = 0$. Suppose that $U'[K] - V_x + [U, V] = 0$ and $U'[S] - W_x + [U, W] = 0$, we define

$$[\![V, W]\!]_0 = V'[S] - W'[K] + [V, W] \tag{2.11}$$

which is well defined. This moment $\mathcal{M}(U,0)$ constitutes an algebra with regard to $[\![\cdot,\cdot]\!]_0$ and thus $\langle E\mathcal{M}(U,0), [\cdot,\cdot]\rangle$ is a Lie algebra. Set $K(U) = \{V \in \widetilde{\mathcal{V}}^r_{(0)} | V_x = [U, V]\}$. Obviously $K(U)$ is a subspace of $\mathcal{M}(U,0)$ and the bracket $[\![\cdot,\cdot]\!]_0$ over $K(U)$ reduces to the matrix commutator $[\cdot,\cdot]$. Moreover by theorem 2.1, we may see that $[\![K(U), \mathcal{M}(U,0)]\!]_0$, $[\![\mathcal{M}(U,0), K(U)]\!]_0 \subseteq K(U)$. Therefore $K(U)$ is an ideal subalgebra of $\langle \mathcal{M}(U,0), [\![\cdot,\cdot]\!]_0\rangle$. In this way, we can generate a quotient algebra $\langle \mathcal{M}(U,0)/K(U), [\![\cdot,\cdot]\!]_0\rangle$. By using corollary 2.1, we may acquire the following result.

Theorem 2.2. The quotient algebra $\langle CL(\mathcal{M}(U,0)) := \mathcal{M}(U,0)/K(U), [\![\cdot,\cdot]\!]_{\bar{0}}\rangle$ is a Lie algebra and isomorphic to the Lie algebra $\langle E\mathcal{M}(U,0), [\cdot,\cdot]\rangle$ under the mapping

$$\rho : CL(\mathcal{M}(U,0)) = \mathcal{M}(U,0)/K(U) \to E\mathcal{M}(U,0)$$

$$CL(V) := \{W \in \mathcal{M}(U,0) \mid W - V \in K(U)\} \mapsto K$$

where $V \in \mathcal{M}(U,0)$, $K \in E\mathcal{M}(U,0)$ satisfy $U'[K] - V_x + [U,V] = 0$.

This theorem also implies that an equation $u_t = K$ $(U'[K] - V_x + [U,V] = 0)$ only possesses a set $CL(V)$ of Lax operators. A similar result for L–A–B triad representations of integrable systems has been established in [13]. In the next section we shall use the above theory of Lax operators to discuss the case of coupled KdV systems. In particular, by using corollary 2.2 of theorem 2.1, we know that we may calculate the algebra relations of the corresponding Lax operators to give the symmetry algebras of integrable systems.

3. The non-isospectral flows and Lax operator algebra of coupled KdV systems

Let us now consider the spectral problem (1.1) with the potential $u = (v_0, v_1, \ldots, v_{q-1})^T$. Setting $\psi = \phi_1$, $\psi_x = \phi_2$, then (1.1) may be rewritten as

$$\phi_x = U\phi \qquad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \qquad U = \begin{bmatrix} 0 & 1 \\ -Q & 0 \end{bmatrix}. \tag{3.1}$$

Suppose that the associated problem is as follows

$$\phi_t = V\phi \qquad V = \begin{bmatrix} V^{(1)} & V^{(2)} \\ V^{(3)} & -V^{(1)} \end{bmatrix} \in \widetilde{\mathcal{V}}^2_{(0)}.$$

Clearly the compatibility condition $U_t - V_x + [U,V] = 0$ in this case gives equivalently

$$V^{(1)} = -\tfrac{1}{2}V_x^{(2)} \qquad V^{(3)} = V_x^{(1)} - QV^{(2)} \qquad Q_t = -V_x^{(3)} - 2QV^{(1)}.$$

Therefore, we have

$$Q_t = 2D[(\tfrac{1}{4}D^2 + Q - \tfrac{1}{2}IQ_x)V^{(2)}]$$

where $D = \partial/\partial x$, $I = \int_{-\infty}^{x} \mathrm{d}x'$, $DI = ID = 1$. We make

$$R_i = \tfrac{1}{4}\delta_{i l}D^2 + v_i - \tfrac{1}{2}Iv_{ix} \qquad 0 \leqslant i \leqslant q. \tag{3.2}$$

Note that $R_q = -1$. Eventually we arrive at

$$\sum_{i=0}^{q-1} v_{it}\lambda^i + \sum_{i=0}^{q}(i-l)v_i\lambda^{i-1}\lambda_t = 2D\sum_{i=0}^{q} R_i\lambda^i V^{(2)}. \tag{3.3}$$

3.1. The isospectral ($\lambda_t = 0$) case

We choose that

$$a_j = 0 \qquad j < 0 \qquad a_0 = 1 \qquad a_j = \sum_{i=0}^{q-1} R_i a_{i+j-q} \qquad j \geqslant 1. \quad (3.4)$$

Then we set

$$V_m = \begin{bmatrix} V_m^{(1)} & V_m^{(2)} \\ V_m^{(3)} & -V_m^{(1)} \end{bmatrix} \qquad V_m^{(2)} = \sum_{j=0}^{m} a_j \lambda^{m-j} \qquad m \geqslant 0. \quad (3.5)$$

$$V_m^{(1)} = -\tfrac{1}{2} V_{mx}^{(2)} \qquad V_m^{(3)} = -\tfrac{1}{2} V_{mxx}^{(2)} - Q V_m^{(2)}$$

Because $R_q = -1$, we have

$$\sum_{j=0}^{q} R_j a_{j+m-i} = \begin{cases} -1 & i = q+m \\ 0 & q \leqslant i \leqslant q+m-1 \end{cases}$$

which is equivalent to (3.4) in fact. In this way, we see that

$$2D \sum_{i=0}^{q} R_i \lambda^i V_m^{(2)} = 2D \sum_{i=0}^{q-1} \left(\sum_{j=0}^{i} R_j a_{j+m-i} \right) \lambda^i + 2D \sum_{i=q}^{q+m} \left(\sum_{j=0}^{q} R_j a_{j+m-i} \right) \lambda^i$$

$$= 2D \sum_{i=0}^{q-1} \left(\sum_{j=0}^{i} R_j a_{j+m-i} \right) \lambda^i.$$

Hence the equality (3.3) gives an isospectral ($\lambda_t = 0$) hierarchy of evolution equations

$$u_t = K_m = 2D \left(R_0 a_m, R_0 a_{m-1} + R_1 a_m, \ldots, \sum_{j=0}^{q-2} R_j a_{j+m-q+2}, a_{m+1} \right)^T \qquad m \geqslant 0.$$

From the above deduction, we know that $\{a_j\}_{j=0}^{\infty}$ in (3.5) is unique except a constant multiple when the integral constants are selected as zero. Let

$$\Phi = \begin{bmatrix} 0 & 0 & \cdots & 0 & R_0^* \\ 1 & 0 & \cdots & 0 & R_1^* \\ 0 & 1 & \cdots & 0 & R_2^* \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & R_{q-1}^* \end{bmatrix} \quad (3.6)$$

with $R_i^* = \frac{1}{4}\delta_{il} D^2 + v_i + \frac{1}{2} v_{ix} I$, $0 \leqslant i \leqslant q-1$, which are the conjugate operators of R_i, $0 \leqslant i \leqslant q-1$. Here Φ has been proved to be a hereditary symmetry (see [9–11]). Since $DR_i = R_i^* D$, $0 \leqslant i \leqslant q-1$, the isospectral hierarchy becomes

$$u_t = K_m = \Phi K_{m-1} = \cdots = \Phi^m K_0 = \Phi^m u_x \qquad m \geqslant 0 \quad (3.7)$$

by which Φ is also a common strong symmetry of the whole hierarchy according to the result from Fuchssteiner [8]. It is clear that the coupled KdV hierarchy (3.7) with $q = 1$ (simultaneously $l = 0$) is just the KdV hierarchy of integrable systems (see [14] for example). For this coupled KdV hierarchy (3.7), many integrable properties such as multi-Hamiltonian structures, Miura maps, modified systems and compatible Poisson brackets have been presented in [9–11, 15, 16].

3.2. The non-isospectral ($\lambda_t = \lambda^{n+1}$, $n \geqslant 0$) case

In this case, we choose that

$$b_j = 0 \qquad j < 0 \qquad b_0 = \tfrac{1}{2}(q-l)x$$
$$b_j = \tfrac{1}{2}(j-q+l)Iv_{q-j} + \sum_{i=0}^{q-1} R_i b_{i+j-q} \qquad j \geqslant 1 \tag{3.8}$$

where $v_i = 0$, $i < 0$. Then we set

$$W_n = \begin{bmatrix} W_n^{(1)} & W_n^{(2)} \\ W_n^{(3)} & -W_n^{(1)} \end{bmatrix} \qquad W_n^{(2)} = \sum_{j=0}^{n} b_j \lambda^{n-j} \qquad n \geqslant 0. \tag{3.9}$$
$$W_n^{(1)} = -\tfrac{1}{2} W_{nx}^{(2)} \qquad W_n^{(3)} = -\tfrac{1}{2} W_{nxx}^{(2)} - Q W_n^{(2)}$$

This moment (3.3) becomes

$$\sum_{i=0}^{q-1} v_{it}\lambda^i + \sum_{i=0}^{q-1}(i-l-n)v_{i-n}\lambda^i + \sum_{i=q}^{q+n}(i-l-n)v_{i-n}\lambda^i$$
$$= 2D\sum_{i=0}^{q-1}\left(\sum_{j=0}^{i} R_j b_{j+n-i}\right)\lambda^i + 2D\sum_{i=q}^{q+n}\left(\sum_{j=0}^{q} R_j b_{j+n-i}\right)\lambda^i.$$

Since we have

$$(i-l-n)v_{i-n} = 2D\sum_{j=0}^{q} R_j b_{j+n-i} \qquad q \leqslant i \leqslant q+n$$

equivalent to (3.8) in fact, the non-isospectral hierarchy of flows reads as

$$u_t = \sigma_n = 2D\Bigg(\tfrac{1}{2}(l+n)Iv_{-n} + R_0 b_n,\ \tfrac{1}{2}(l+n-1)Iv_{1-n} + R_0 b_{n-1}$$
$$+ R_1 b_n, \ldots, \tfrac{1}{2}(l+n-q+2)Iv_{q-2-n} + \sum_{j=0}^{q-2} R_j b_{j+n-q+2},\ b_{n+1}\Bigg)^T \qquad n \geqslant 0.$$

Further we may obtain

$$u_t = \sigma_n = \Phi\sigma_{n-1} = \cdots = \Phi^n \sigma_0 \qquad n \geqslant 0 \tag{3.10a}$$

with

$$\sigma_0 = (\,qv_0 + \tfrac{1}{2}(q-l)xv_{0x}, (q-1)v_1 + \tfrac{1}{2}(q-l)xv_{1x}, \ldots, v_{q-1} + \tfrac{1}{2}(q-l)xv_{q-1,x})^T. \tag{3.10b}$$

Below we want to verify that two hierarchies of Lax operators $\{V_m\}_{m=0}^{\infty}$, $\{W_n\}_{n=0}^{\infty}$ constitute an infinite-dimensional Lie algebra with regard to the product (2.7) and thus we will generate the symmetry algebra of coupled KdV systems (3.7). Let

$$\mathcal{B}(\lambda) = \left\{ B = \sum_{j=0}^{n} a_j \lambda^{n-j} \,|\, n \geqslant 0,\ a_j \in \mathcal{B},\ 0 \leqslant j \leqslant n \right\}$$
$$\mathcal{Z}_{\mathrm{CKdV}} = \left\{ V = \begin{bmatrix} V^{(1)} & V^{(2)} \\ V^{(3)} & -V^{(1)} \end{bmatrix} \in \widetilde{\mathcal{V}}^2_{(0)} \,|\, V^{(2)} \in \mathcal{B}(\lambda) \right\}$$

Proposition 3.1. Let U be given by (3.1). If $V \in \mathcal{Z}_{\mathrm{CKdV}} \cap \mathcal{M}(U, 0)$ and $V|_{u=0} = 0$, then $V = 0$.

Proof. Assume that

$$V=\begin{bmatrix} V^{(1)} & V^{(2)} \\ V^{(3)} & -V^{(1)} \end{bmatrix}=\begin{bmatrix} V^{(1)} & \sum_{j=0}^{n} a_j\lambda^{n-j} \\ V^{(3)} & -V^{(1)} \end{bmatrix}\in \mathcal{Z}_{\rm CKdV}$$

and $K=(K_1,\ldots,K_q)^T\in\mathcal{B}^q$ such that $U'[K]-V_x+[U,V]=0$. Analogously to the deduction of the isospectral hierarchy (3.7), we may obtain

$$V^{(1)}=-\tfrac{1}{2}V_x^{(2)}\qquad V^{(3)}=-\tfrac{1}{2}V_{xx}^{(2)}-QV^{(2)} \tag{3.11}$$

$$\sum_{i=0}^{q-1}K_{i+1}\lambda^i=2D\sum_{i=0}^{q-1}\left(\sum_{j=0}^{i}R_ja_{j+n-i}\right)\lambda^i+2D\sum_{i=q}^{q+n}\left(\sum_{j=0}^{q}R_ja_{j+n-i}\right)\lambda^i.$$

Therefore we have

$$D\sum_{j=0}^{q}R_ja_{j+n-i}=0\qquad q\leqslant i\leqslant q+n.$$

It follows that $V^{(2)}=0$ since $R_q=-1$ and $V^{(2)}|_{u=0}=0$. Also by (3.11), $V^{(1)}=V^{(3)}=0$. In this way, $V=0$. □

Denote by P_{12} the projection over $\widetilde{\mathcal{V}}^2_{(0)}$:

$$P_{12}:W=\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}\mapsto W_{12}\qquad W\in\widetilde{\mathcal{V}}^2_{(0)}. \tag{3.12}$$

For any

$$V=\begin{bmatrix} V^{(1)} & V^{(2)} \\ V^{(3)} & -V^{(1)} \end{bmatrix}\qquad W=\begin{bmatrix} W^{(1)} & W^{(2)} \\ W^{(3)} & -W^{(1)} \end{bmatrix}\in\widetilde{\mathcal{V}}^2_{(0)}\qquad K,S\in\mathcal{B}^q,\ f,g\in C^\infty(C)$$

if (V,K,f), $(W,S,g)\in\mathcal{P}(U)$, then we can find

$$P_{12}([\![V,W]\!])=V^{(2)\prime}[S]-W^{(2)\prime}[K]+V^{(2)}W_x^{(2)}-V_x^{(2)}W^{(2)}+gV_\lambda^{(2)}-fW_\lambda^{(2)}. \tag{3.13}$$

Therefore we can easily obtain

Proposition 3.2. The functions $P_{12}([\![V_m,V_n]\!])$, $P_{12}([\![V_m,W_n]\!])$, $P_{12}([\![W_m,W_n]\!])$, $m,n\geqslant 0$, all belong to the space $\mathcal{B}(\lambda)$ and thus the operators $[\![V_m,V_n]\!]$, $[\![V_m,W_n]\!]$, $[\![W_m,W_n]\!]$, $m,n\geqslant 0$, are all among the space $\mathcal{Z}_{\rm CKdV}$.

It is easy to see that for $m,n\geqslant 0$,

$$K_m|_{u=0}=\sigma_n|_{u=0}=0\qquad Q|_{u=0}=-\lambda^{q-l}$$

$$V_m|_{u=0}=\begin{bmatrix} 0 & 1 \\ \lambda^{q-l} & 0 \end{bmatrix}\lambda^m\qquad V_{m\lambda}|_{u=0}=(V_m|_{u=0})_\lambda=\begin{bmatrix} 0 & m \\ (q-l+m)\lambda^{q-l} & 0 \end{bmatrix}\lambda^{m-1}$$

$$W_n|_{u=0}=\frac{1}{4}(q-l)\begin{bmatrix} -1 & 2x \\ 2x\lambda^{q-l} & 1 \end{bmatrix}\lambda^n$$

$$W_{n\lambda}|_{u=0}=(W_n|_{u=0})_\lambda=\frac{1}{4}(q-l)\begin{bmatrix} -n & 2nx \\ 2(q-l+n)x\lambda^{q-l} & n \end{bmatrix}\lambda^{n-1}.$$

Proposition 3.3. For $m, n \geqslant 0$, we have

$$[\![V_m, V_n]\!]|_{u=0} = 0 \qquad m, n \geqslant 0$$

$$[\![V_m, W_n]\!]|_{u=0} = [\tfrac{1}{2}(q-l) + m]V_{m+n}|_{u=0} \qquad m, n \geqslant 0$$

$$[\![W_m, W_n]\!]|_{u=0} = (m-n)W_{m+n}|_{u=0} \qquad m, n \geqslant 0.$$

Proof. Let $m, n \geqslant 0$. We can calculate that

$$\begin{aligned}[\![V_m, W_n]\!]|_{u=0} &= [\![V_m|_{u=0}, W_n|_{u=0}]\!] = [V_m|_{u=0}, W_n|_{u=0}] + \lambda^{n+1}V_{m\lambda}|_{u=0} \\ &= \begin{bmatrix} 0 & \frac{1}{2}(q-l) \\ -\frac{1}{2}(q-l)\lambda^{q-l} & 0 \end{bmatrix}\lambda^{m+n} + \begin{bmatrix} 0 & m \\ (q-l+m)\lambda^{q-l} & 0 \end{bmatrix}\lambda^{m+n} \\ &= [\tfrac{1}{2}(q-l) + m]V_{m+n}|_{u=0}.\end{aligned}$$

The rest may be proved similarly. □

Now we give the concrete form of Lax operator algebra of coupled KdV systems.

Theorem 3.1. Let $V_m, W_n, m, n \geqslant 0$, be given by (3.5), (3.9), respectively. Then their commutator relations defined by (2.7) are as follows

$$[\![V_m, V_n]\!] = 0 \qquad m, n \geqslant 0 \tag{3.14a}$$

$$[\![V_m, W_n]\!] = [\tfrac{1}{2}(q-l) + m]V_{m+n} \qquad m, n \geqslant 0 \tag{3.14b}$$

$$[\![W_m, W_n]\!] = (m-n)W_{m+n} \qquad m, n \geqslant 0 \tag{3.14c}$$

which implies that the two hierarchies of Lax operators $\{V_m\}_{m=0}^{\infty}$, $\{W_n\}_{n=0}^{\infty}$ associated with coupled KdV systems constitute an infinite-dimensional Lie algebra under the product (2.7).

Proof. We only verify (3.14b) since the proofs of the relations are completely similar. By theorem 2.1, $V := [\![V_m, V_n]\!] - [\frac{1}{2}(q-l) + m]V_{m+n}$ belongs to $\mathcal{M}(U, 0)$. Besides, we have $V \in \mathcal{Z}_{\mathrm{CKdV}}$ and $V|_{u=0} = 0$ by propositions 3.2 and 3.3, respectively. It follows from the first proposition that V is a zero operator, which is exactly (3.14b). The proof is completed. □

We remark that the Gateaux derivative $U' : \mathcal{B}^q \to \widetilde{\mathcal{V}}^2_{(0)}$ is injective. Therefore we can obtain a Lie algebra of isospectral and non-isospectral vector fields.

Corollary 3.1. Let $K_m, \sigma_n, m, n \geqslant 0$, be given by (3.7), and (3.10), respectively. Then

$$[K_m, K_n] = 0 \qquad m, n \geqslant 0$$

$$[K_m, \sigma_n] = [\tfrac{1}{2}(q-l) + m]K_{m+n} \qquad m, n \geqslant 0$$

$$[\sigma_m, \sigma_n] = (m-n)\sigma_{m+n} \qquad m, n \geqslant 0.$$

From this corollary we can immediately deduce the following symmetry algebra of coupled KdV systems (3.7).

Theorem 3.2. Let K_m, σ_n, $m, n \geqslant 0$, be determined by (3.7) and (3.10), respectively. Then every integrable system $u_t = K_s (s \geqslant 0)$ possesses a hierarchy K-symmetries of $\{K_m\}_{m=0}^{\infty}$ and a hierarchy of τ-symmetries $\tau_n^{(s)} = \{[\frac{1}{2}(q-l)+s]tK_{n+s}+\sigma_n\}_{n=0}^{\infty}$ which constitute an infinite-dimensional Lie algebra with the commutator relations

$$[K_m, K_n] = 0 \qquad m, n \geqslant 0$$
$$[K_m, \tau_n^{(s)}] = [\tfrac{1}{2}(q-l)+m]K_{m+n} \qquad m, n \geqslant 0$$
$$[\tau_m^{(s)}, \tau_n^{(s)}] = (m-n)\tau_{m+n}^{(s)} \qquad m, n \geqslant 0.$$

Theorem 3.2 also shows that $\{\sigma_n\}_{n=0}^{\infty}$ given by (3.10) is common master symmetries of first-order for all coupled KdV systems in (3.7). We point out that the above symmetry algebra may be acquired by using the skeleton of Ma [4] and Oevel [5]. But here it is more natural to generate the symmetry algebra on the basis of zero curvature representations. Moreover, the Lax operator algebra hidden in the back of the symmetry algebra is exposed. In fact, it is because isospectral ($\lambda_t = 0$) and non-isospectral ($\lambda_t = \lambda^n$, $n \geqslant 0$) Lax operators constitute a Lie algebra under the product (2.7) introduced in section 2 that isospectral integrable systems possess a symmetry algebra. We will give a fuller description of this property related to zero curvature representations in a future publication.

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